

Index Assignment for Quantized Beamforming MIMO Systems

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Abstract—Index assignment (IA) technique is introduced to quantized beamforming systems. The feedback channel in these systems is modeled as a discrete memoryless channel (DMC), and the diversity and array gains with feedback errors are derived. Based on the analytical results, IA scheme is designed to provide a redundancy-free protection against feedback errors. Simulation results show that good IA scheme improves the array gain and symbol error rate (SER).

Index Terms—Beamforming, diversity methods, MIMO systems, finite-rate feedback, index assignment.

I. INTRODUCTION

MULTIPLE-INPUT multiple-output (MIMO) wireless systems have received much interest due to their high capacity and reliability. Transmit beamforming with receive combining is important in the research of MIMO systems [1], because it provides large diversity and array gains without sophisticated signal processing at the transmitter and receiver. However, in a beamforming system, the transmitter requires the channel state information, which is not usually available. This motivates recent researches on beamforming with quantized feedback.

The design and performance analysis of quantized beamforming systems have been extensively studied, e.g. [2]–[7]. In [2][3], bounds on outage probability and average receive signal-to-noise ratio (SNR) were derived, and good beamformers were designed to approach these bounds. The beamformer design was treated as a generalized vector quantization (VQ) problem in [6], and a generalized Lloyd algorithm was employed to construct codebooks. Analytical results showed that quantization errors cause array gain loss [5][7], but don't affect the diversity gain [4].

In [2]–[7], the feedback was assumed to be error-free, but feedback errors are present in practice. In this paper, we study a quantized beamforming MIMO system with feedback errors. It is shown that feedback errors deteriorate not only the array gain but also the diversity gain. To mitigate the effect of feedback errors, IA technique is adopted, which is popular in conventional VQ designs (see [8][9] and references therein). IA technique is preferable to other error-protection methods, e.g. error-control coding, because it requires neither additional feedback bits nor additional signal processing, i.e. it is *redundancy-free*.

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II. SYSTEM MODEL

Fig.1 shows a frequency-flat MIMO system with N_t transmit and N_r receive antennas.

A. Forward Part. The forward channel is modeled as a $N_r \times N_t$ random matrix $\mathbf{H}$, which has independent $\mathcal{CN}(0, 1)$ entries, remains static within a block, and changes independently between blocks. The system adopts transmit beamforming with receive combining. With $s \in \mathbb{C}$ denoting the transmitted symbol, the signal $r \in \mathbb{C}$ after receive combining is given by

$$r = \mathbf{z}^H \mathbf{H} \mathbf{w} s + \mathbf{z}^H \boldsymbol{\eta}, \quad (1)$$

where $\boldsymbol{\eta} \in \mathbb{C}^{N_r}$ is the noise vector with independent $\mathcal{CN}(0, N_0)$ entries; $\mathbf{w} \in \mathbb{C}^{N_t}$ is the unit-norm beamforming vector; $\mathbf{z} \in \mathbb{C}^{N_r}$ is the unit-norm combining vector. Assuming that the receiver knows the channel $\mathbf{H}$ and the beamforming vector $\mathbf{w}$, the maximum ratio combining (MRC) vector is $\mathbf{z} = \mathbf{H} \mathbf{w} / \|\mathbf{H} \mathbf{w}\|$ [3]. And the corresponding instantaneous receive SNR is

$$\gamma = \gamma_0 \|\mathbf{H} \mathbf{w}\|^2, \quad (2)$$

where $\|\cdot\|$ denotes the 2-norm of a vector; and $\gamma_0 = \mathbb{E}(|s|^2)/N_0$ is the SNR.

B. Feedback part. In a quantized beamforming system, a codebook $\mathcal{C}$, which has $K = 2^B$ unit-norm codewords $\mathcal{C} = \{\mathbf{c}_k\}_{k=1}^K$, is designed in advance and known to both the transmitter and the receiver. This codebook can be constructed using the methods in [2][3][6]. The receiver feeds back an index $k \in \{1, \dots, K\}$ to the transmitter, requiring B bits each time.

In Fig.1, the feedback part consists of five modules. The *VQ encoder module* finds out the optimal codeword by maximizing the instantaneous receive SNR, i.e.

$$\mathbf{c}_{\text{op}} = \arg \max_{\mathbf{c} \in \mathcal{C}} \|\mathbf{H} \mathbf{c}\|^2. \quad (3)$$

The VQ encoder outputs the index of $\mathbf{c}_{\text{op}}$, say k , to the *IA module*, which performs permutation¹ Π on this index. The permuted index $\Pi(k)$ is sent to the transmitter via the *discrete memoryless channel (DMC) module* with transition probability

$$t(i, j) = \mathbb{P}(\text{DMC output } j \mid \text{DMC input } i), \\ i, j = 1, \dots, K. \quad (4)$$

Suppose that the output of the DMC is $\Pi(l)$. The *inverse IA module* performs the inverse-permutation Π^{-1} on $\Pi(l)$, and outputs l . The *VQ decoder module* selects the codeword $\mathbf{c}_l$ from the codebook $\mathcal{C}$. And the transmitter will use $\mathbf{c}_l$ as the beamforming vector, i.e. $\mathbf{w} = \mathbf{c}_l$.

¹A permutation Π is an invertible operator from the index set $\mathcal{I} = \{1, \dots, K\}$ onto itself. For each index $k \in \mathcal{I}$, Π uniquely maps k to another index $\Pi(k) \in \mathcal{I}$. There are $K!$ possible permutations on $\mathcal{I}$. Each permutation represents an IA scheme.

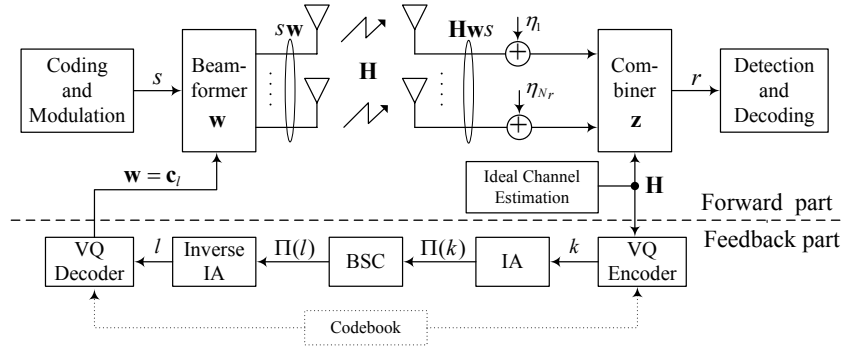


Fig. 1. The system model.

From the above description, we know that conditioned on the optimal codeword $\mathbf{c}_{\text{op}} = \mathbf{c}_k$, the probability that the transmitter uses $\mathbf{c}_l$ as the beamforming vector is

$$\mathbb{P}(\mathbf{w} = \mathbf{c}_l | \mathbf{c}_{\text{op}} = \mathbf{c}_k) = \mathbb{P}(\text{DMC output } \Pi(l) | \text{DMC input } \Pi(k)) = t(\Pi(k), \Pi(l)), \quad k, l = 1, \dots, K. \quad (5)$$

III. THE DIVERSITY AND ARRAY GAINS

A. Diversity Gain. Diversity gain refers to the slope of SER-vs-SNR curve (on a log-log scale) as SNR approaches infinity. With error-free feedback, a well-designed beamformer can provide full diversity gain $N_t \times N_r$ if the codebook cardinality $K \geq N_t$ [4]. However, the diversity gain may decrease to N_r due to feedback errors, as shown in the following theorem.

Theorem 1 (the diversity gain). For the quantized beamforming system shown in Sec.II, the diversity gain equals to N_r , if the transition probability of the DMC feedback channel satisfies

$$t(i, j) \geq t_{\min} > 0, \quad i, j = 1, \dots, K. \quad (6)$$

Proof: See Appendix A.

Remark1 (BSC). The constraint (6) is satisfied by many DMC's. For example, binary symmetric channel (BSC) is an important DMC, whose transition probability is

$$t^{\text{BSC}}(i, j) = p^{d_H(i-1, j-1)} (1-p)^{B-d_H(i-1, j-1)}, \quad i, j = 1, \dots, K, \quad (7)$$

where p is a parameter of the BSC; $K = 2^B$; and $d_H(i-1, j-1)$ denotes the Hamming distance between the binary representations of $i-1$ and $j-1$. The BSC satisfies (6) if $p > 0$. Hence, the diversity gain of a quantized beamforming system is N_r , if the feedback channel is a BSC.

Remark2 (comparison with STBC). With error-free feedback, quantized beamforming outperforms space-time block coding (STBC), because quantized beamforming provides not only diversity gain but also array gain. However, this conclusion should be reconsidered if there are feedback errors. According to Theorem 1, a quantized beamforming system may suffer a large diversity gain loss due to feedback errors. So we can predict that at sufficiently high SNR, the SER of quantized beamforming will be worse than that of STBC. The comparison at low-to-medium SNR is of practical importance, but out of the scope of this paper.

B. Array Gain. The array gain is defined as the ratio of the average receive SNR $\mathbb{E}(\gamma)$ and the SNR γ_0 . It reflects the increase in average receive SNR that arises from the coherent combining effect of multiple antennas.

Consider the case that the VQ encoder selects $\mathbf{c}_k$ as the optimal codeword, but the beamformer uses $\mathbf{c}_l$ as the beamforming vector due to feedback errors. From (2), the average receive SNR conditioned on this case is $\gamma_0 \mathbb{E}(\|\mathbf{H}\mathbf{c}_l\|^2 | \mathbf{c}_{\text{op}} = \mathbf{c}_k) = \gamma_0 \mathbf{c}_l^H \mathbb{E}(\mathbf{H}^H \mathbf{H} | \mathbf{c}_{\text{op}} = \mathbf{c}_k) \mathbf{c}_l$. Therefore, by the law of total probability, the array gain is written as

$$\begin{aligned} \mathbb{E}(\gamma)/\gamma_0 &= \sum_{k=1}^K \sum_{l=1}^K \mathbf{c}_l^H \mathbb{E}(\mathbf{H}^H \mathbf{H} | \mathbf{c}_{\text{op}} = \mathbf{c}_k) \mathbf{c}_l \\ &\quad \times \mathbb{P}(\mathbf{w} = \mathbf{c}_l | \mathbf{c}_{\text{op}} = \mathbf{c}_k) \mathbb{P}(\mathbf{c}_{\text{op}} = \mathbf{c}_k) \\ &= \sum_{k=1}^K \sum_{l=1}^K \mathbf{c}_l^H \mathbb{E}(\mathbf{H}^H \mathbf{H} | \mathbf{c}_{\text{op}} = \mathbf{c}_k) \mathbf{c}_l \\ &\quad \times t(\Pi(k), \Pi(l)) \mathbb{P}(\mathbf{c}_{\text{op}} = \mathbf{c}_k), \end{aligned} \quad (8)$$

where the second equation follows from (5). In (8), the accurate values of $\mathbb{E}(\mathbf{H}^H \mathbf{H} | \mathbf{c}_{\text{op}} = \mathbf{c}_k)$ and $\mathbb{P}(\mathbf{c}_{\text{op}} = \mathbf{c}_k)$ are hard to obtain, but their approximations can be derived as follows.

Since the channel matrix $\mathbf{H}$ has independent $\mathcal{CN}(0, 1)$ entries, $\mathbf{H}^H \mathbf{H}$ is Wishart distributed. Its eigen-decomposition is denoted as

$$\mathbf{H}^H \mathbf{H} = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^H = \sum_{n=1}^{N_t} \lambda_n \mathbf{u}_n \mathbf{u}_n^H, \quad (9)$$

where $\mathbf{\Lambda} = \text{diag}\{\lambda_1, \dots, \lambda_{N_t}\}$ and $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_{N_t}]$; $\lambda_1 \geq \dots \geq \lambda_{N_t} \geq 0$ are eigenvalues of $\mathbf{H}^H \mathbf{H}$; $\mathbf{u}_n, n = 1, \dots, N_t$, is the eigenvector corresponding to λ_n . The distribution of nonzero eigenvalues is known. The eigen matrix $\mathbf{U}$ is uniformly distributed on the group of $N_t \times N_t$ unitary matrices and *independent of the eigenvalues* [3, Lemma 1].

If the feedback channel is perfect, the transmitter uses the eigenvector $\mathbf{u}_1$ as the beamforming vector, which is called maximum ratio transmission (MRT). But this is not the case in practice, where the quantized information — the optimal codeword $\mathbf{c}_{\text{op}}$ — is fed back to the transmitter. Then, one would expect that the ideal feedback information $\mathbf{u}_1$ and the quantized version $\mathbf{c}_{\text{op}}$ are 'close'. Since both $\mathbf{u}_1$ and $\mathbf{c}_{\text{op}}$ belong to the unit hypersphere $\Omega^{N_t} = \{\mathbf{x} \in \mathbb{C}^{N_t} : \mathbf{x}^H \mathbf{x} = 1\}$, a suitable measure of their 'closeness' is the chordal distance, defined

as [3]

$$d_c(\mathbf{x}_1, \mathbf{x}_2) = \sqrt{1 - |\mathbf{x}_1^H \mathbf{x}_2|^2}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \Omega^{N_t}. \quad (10)$$

Now define a spherical cap centered at the codeword $\mathbf{c}_k$ as

$$\begin{aligned} \mathcal{S}(\mathbf{c}_k) &= \{\mathbf{x} \in \Omega^{N_t} : d_c(\mathbf{x}, \mathbf{c}_k) < \sqrt{\alpha}\} \\ &= \{\mathbf{x} \in \Omega^{N_t} : |\mathbf{x}^H \mathbf{c}_k|^2 > 1 - \alpha\}, \quad k = 1, \dots, K. \end{aligned} \quad (11)$$

We use the approximations: **AP1** $\mathbb{E}(\mathbf{H}^H \mathbf{H} | \mathbf{c}_{\text{op}} = \mathbf{c}_k) \approx \mathbb{E}(\mathbf{H}^H \mathbf{H} | \mathbf{u}_1 \in \mathcal{S}(\mathbf{c}_k))$, $k = 1, \dots, K$, and **AP2** $\mathbb{P}(\mathbf{c}_{\text{op}} = \mathbf{c}_k) \approx \mathbb{P}(\mathbf{u}_1 \in \mathcal{S}(\mathbf{c}_k))$, $k = 1, \dots, K$. These approximations are also used in [7].

The right-hand side (RHS) of **AP1** can be derived as follows

$$\mathbb{E}(\mathbf{H}^H \mathbf{H} | \mathbf{c}_{\text{op}} = \mathbf{c}_k) \approx \mathbb{E}(\mathbf{H}^H \mathbf{H} | \mathbf{u}_1 \in \mathcal{S}(\mathbf{c}_k)) \quad (12)$$

$$= \sum_{n=1}^{N_t} \mathbb{E}(\lambda_n) \mathbb{E}(\mathbf{u}_n \mathbf{u}_n^H | \mathbf{u}_1 \in \mathcal{S}(\mathbf{c}_k)) \quad (13)$$

$$= \frac{N_t(1-\alpha)}{N_t-1} (\mathbb{E}(\lambda_1) - N_r) \mathbf{c}_k \mathbf{c}_k^H + \left(\frac{N_r(N_t-\alpha)}{N_t-1} - \frac{1-\alpha}{N_t-1} \mathbb{E}(\lambda_1) \right) \mathbf{I}_{N_t}. \quad (14)$$

The result in (13) follows from (9); and (14) results from (24) (25) in Appendix B and the fact $\sum_{n=1}^{N_t} \mathbb{E}(\lambda_n) = \mathbb{E}(\|\mathbf{H}\|_F^2) = N_t N_r$. According to [2], the RHS of **AP2** reduces to

$$\mathbb{P}(\mathbf{c}_{\text{op}} = \mathbf{c}_k) \approx \mathbb{P}(\mathbf{u}_1 \in \mathcal{S}(\mathbf{c}_k)) = \alpha^{N_t-1}, \quad k = 1, \dots, K. \quad (15)$$

The calculation is straightforward, and omitted here. To consist with $\sum_{k=1}^K \mathbb{P}(\mathbf{c}_{\text{op}} = \mathbf{c}_k) = 1$, we set $\alpha = \left(\frac{1}{K}\right)^{\frac{1}{N_t-1}}$. Substituting (14) and (15) into (8), the following proposition is obtained.

Proposition 1 (the array gain). For the quantized beamforming system shown in Sec.II, the array gain is approximated by

$$\begin{aligned} \mathbb{E}(\gamma)/\gamma_0 &\approx \frac{N_r(N_t-\alpha)}{N_t-1} - \frac{1-\alpha}{N_t-1} \mathbb{E}(\lambda_1) + \frac{N_t(1-\alpha)}{K(N_t-1)} (\mathbb{E}(\lambda_1) - N_r) \\ &\quad \times \sum_{k=1}^K \sum_{l=1}^K \{ |\mathbf{c}_k^H \mathbf{c}_l|^2 t(\Pi(k), \Pi(l)) \}, \end{aligned} \quad (16)$$

where $\alpha = \left(\frac{1}{K}\right)^{\frac{1}{N_t-1}}$; $\mathbf{c}_k, k = 1, \dots, K$, is the codeword; Π represents the IA scheme; and λ_1 denotes the largest eigenvalue of $\mathbf{H}^H \mathbf{H}$.

In general, $\mathbb{E}(\lambda_1)$ can be obtained by numerical integration or simulation. It has a closed-form expression in some cases. If $\min(N_t, N_r) = 1$, $\mathbf{H}$ reduces to a vector $\mathbf{h}$, and $\mathbb{E}(\lambda_1) = \mathbb{E}(\|\mathbf{h}\|^2) = \max(N_t, N_r)$. If $\min(N_t, N_r) = 2$, a closed form expression of $\mathbb{E}(\lambda_1)$ is derived in [1].

For the case of error-free feedback $t(i, j) = \delta_{i,j}$, (16) reduces to the result in [7, Sec.III].

IV. THE INDEX ASSIGNMENT

In this section, we focus on the IA design of the quantized beamforming system, using the array gain (or equivalently, the average receive SNR) as a design metric.

According to (16), an IA scheme Π that maximizes the array gain can be obtained by solving

$$\begin{aligned} \max \quad & \sum_{k=1}^K \sum_{l=1}^K \{ |\mathbf{c}_k^H \mathbf{c}_l|^2 \cdot t(\Pi(k), \Pi(l)) \} \\ \text{s.t.} \quad & \Pi \text{ is a permutation.} \end{aligned} \quad (17)$$

Maximization of the cost function in (17) over all of the $K!$ possible permutations is a special case of the quadratic assignment problem(QAP), and is known to be NP-complete. If K is small, it can be solved by brute-force search. If K is large, e.g. $K \geq 16$, brute-force search is prohibitive since $16! > 10^{13}$, and suboptimal methods have been proposed in the literature [8][9].

Remark3 (simplification for BSC). In practice, feedback errors seldom occur, and the effect of multiple bit errors can be neglected. So the transition probability in (7) can be approximated by

$$t^{\text{BSC}}(i, j) \approx t^{\text{Ap}}(i, j) = \begin{cases} (1-p)^B, & d_H(i-1, j-1) = 0; \\ p(1-p)^{B-1}, & d_H(i-1, j-1) = 1; \\ 0, & d_H(i-1, j-1) \geq 2. \end{cases}$$

and (17) is simplified to

$$\begin{aligned} \max \quad & \sum_{k=1}^K \sum_{l=1}^K \{ |\mathbf{c}_k^H \mathbf{c}_l|^2 \cdot t^{\text{Ap}}(\Pi(k), \Pi(l)) \} \\ \text{s.t.} \quad & \Pi \text{ is a permutation.} \end{aligned} \quad (18)$$

The advantage of (18) is two-fold. 1) The computational complexity of the cost function is reduced, since most of t^{Ap} 's are zero. 2) Its solution doesn't depend on the parameter p of the BSC [8]. Once we solve (18) for a particular value of p , the solution is valid for other values.

V. SIMULATION RESULTS

Simulations are carried out in (2,1,8) and (4,2,64) systems, where (N_t, N_r, K) denotes a system with N_t transmit antennas, N_r receive antennas, and codebook cardinality K . In the simulations, the forward channel $\mathbf{H}$ is i.i.d. block-fading and the feedback channel is the BSC in (7). Codebooks are downloaded from [11]. We design good IA's for these codebooks by solving the simplified problem (18). To study the worst-case performance, bad IA's are also designed by minimizing the cost function of (18). The IA's of the (2,1,8) system are obtained by brute-force search, and shown in TABLE I. The IA's of the (4,2,64) system are designed using the binary switching algorithm [8].

From the IA results in TABLE I, we can gain an insight into how the good IA improves the system performance. For example, the first codeword is closest (with respect to the chordal distance) to the last codeword. The Hamming distance between their original indexes (1 and 8 respectively) is 3, while the Hamming distance between their good indexes (7 and 5 respectively) is 1. Similarly, the second codeword is closest to the fifth codeword. The Hamming distance between their original indexes (2 and 5) is 2, and that between their good indexes (1 and 2) is 1. It is shown from these examples that the good IA scheme has assigned close indexes (in term of Hamming distance) to close codewords (in term of chordal distance).

Fig.2 and 3 show the variation of the array gain with the increase of p . In the simulations, the SNR is fixed at $\gamma_0 = 10$ dB. As shown in both figures, the good IA always outperforms the bad IA and the original IA. Furthermore, we can see that the analytical result (16) is tight.

Fig.4 and 5 depict the SER simulation results. QPSK and 16QAM modulations are used in (2,1,8) and (4,2,64)

TABLE I
Index assignment schemes for a codebook downloaded from [11]

Codeword	Original IA, k	Good IA, $\Pi_{\text{good}}(k)$	Bad IA, $\Pi_{\text{bad}}(k)$
$[0.8393 - j 0.2939, -0.1677 + j 0.4256]^T$	1	7	1
$[-0.3427 + j 0.9161, 0.0498 + j 0.2019]^T$	2	1	4
$[-0.2065 + j 0.3371, 0.9166 + j 0.0600]^T$	3	6	2
$[0.3478 - j 0.3351, 0.2584 + j 0.8366]^T$	4	8	8
$[0.1049 + j 0.6820, 0.6537 + j 0.3106]^T$	5	2	5
$[0.0347 - j 0.2716, 0.0935 - j 0.9572]^T$	6	4	3
$[-0.7457 + j 0.1181, -0.4553 - j 0.4719]^T$	7	3	6
$[-0.7983 + j 0.3232, 0.5000 + j 0.0906]^T$	8	5	7

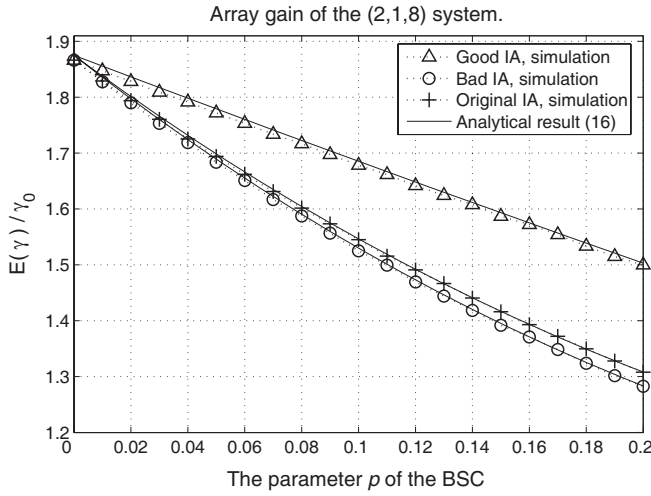


Fig. 2. The array gain of the (2,1,8) system.

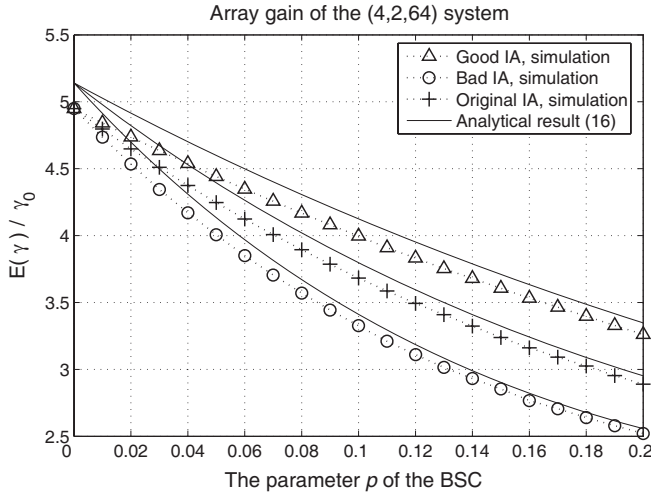


Fig. 3. The array gain of the (4,2,64) system.

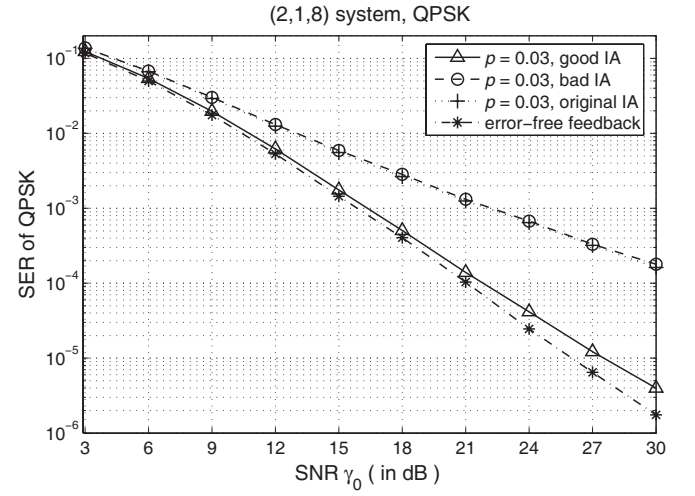


Fig. 4. The SER of QPSK in the (2,1,8) system.

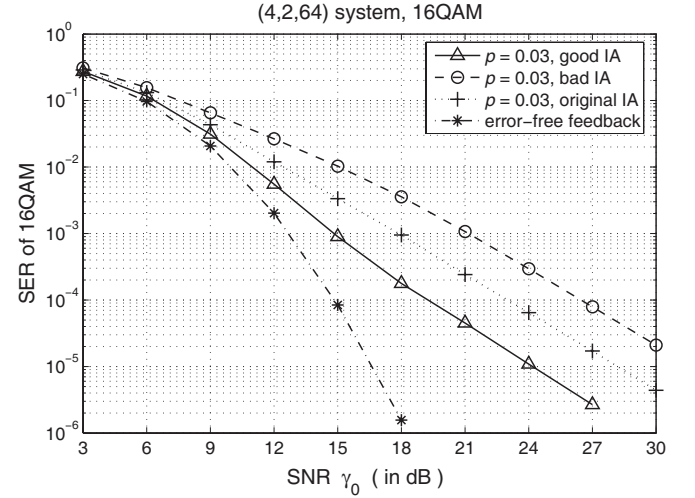


Fig. 5. The SER of 16QAM in the (4,2,64) system.

systems respectively. In the simulations, thirty-six symbols are transmitted in a block, and ideal coherent detection is adopted. In Fig.4, the SER of ‘good IA’ is much lower than that of ‘bad IA’, and approaches that of ‘error-free feedback’. In Fig.5, the good IA outperforms the bad IA, though it cannot reach the performance of error-free feedback. Moreover, the diversity gain (the slope of SER curve at high SNR) fits well with the

conclusion of Theorem 1, except that the diversity gain of (2,1,8) system with good IA is not clear because the SNR is not large enough.

VI. CONCLUSION

In this paper, we studied a quantized beamforming MIMO system with feedback errors. It was observed that the diversity

gain decreases from $N_t N_r$ to N_r due to feedback errors, and the array gain depends on the IA of the codebook. Good IA was designed to maximize the array gain. Simulation results showed that good IA improves the array gain and SER performance.

APPENDIX A PROOF OF THEOREM 1

According to [12], the diversity gain equals to $\lim_{\gamma_0 \rightarrow +\infty} \{-\log P_{\text{out}} / \log \gamma_0\}$, where P_{out} denotes the outage probability. For the quantized beamforming system, P_{out} is given by [2][7]

$$P_{\text{out}} = \mathbb{P}\{\log_2(1 + \gamma_0 \|\mathbf{H}\mathbf{w}\|^2) < R\} \\ = \mathbb{P}\{\|\mathbf{H}\mathbf{w}\|^2 < (2^R - 1)/\gamma_0\}, \quad (19)$$

where γ_0 denotes the SNR; $\mathbf{H}$ is the forward channel matrix; $\mathbf{w}$ is the beamforming vector; and R is the desired transmission rate. By the law of total probability, (19) can be written as

$$P_{\text{out}} = \sum_{k=1}^K \sum_{l=1}^K \mathbb{P}\left(\|\mathbf{H}\mathbf{c}_l\|^2 < \frac{2^R - 1}{\gamma_0} \mid \mathbf{c}_{\text{op}} = \mathbf{c}_k\right) \\ \times \mathbb{P}(\mathbf{w} = \mathbf{c}_l \mid \mathbf{c}_{\text{op}} = \mathbf{c}_k) \mathbb{P}(\mathbf{c}_{\text{op}} = \mathbf{c}_k) \\ = \sum_{k=1}^K \sum_{l=1}^K \mathbb{P}\left(\|\mathbf{H}\mathbf{c}_l\|^2 < \frac{2^R - 1}{\gamma_0} \mid \mathbf{c}_{\text{op}} = \mathbf{c}_k\right) \\ \times t(\Pi(k), \Pi(l)) \mathbb{P}(\mathbf{c}_{\text{op}} = \mathbf{c}_k) \\ \geq t_{\min} \sum_{l=1}^K \left\{ \sum_{k=1}^K \mathbb{P}\left(\|\mathbf{H}\mathbf{c}_l\|^2 < \frac{2^R - 1}{\gamma_0} \mid \mathbf{c}_{\text{op}} = \mathbf{c}_k\right) \right. \\ \left. \times \mathbb{P}(\mathbf{c}_{\text{op}} = \mathbf{c}_k) \right\} \\ = t_{\min} \sum_{l=1}^K \mathbb{P}\left(\|\mathbf{H}\mathbf{c}_l\|^2 < \frac{2^R - 1}{\gamma_0}\right). \quad (20)$$

In above derivation, (5) and (6) are used. Similarly, because $t(\Pi(k), \Pi(l)) \leq 1$ holds trivially, we have

$$P_{\text{out}} \leq \sum_{l=1}^K \mathbb{P}\left(\|\mathbf{H}\mathbf{c}_l\|^2 < \frac{2^R - 1}{\gamma_0}\right). \quad (21)$$

Since the codeword $\mathbf{c}_l$ is deterministic and unit-norm, $\mathbf{H}\mathbf{c}_l \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{N_r})$ and $\|\mathbf{H}\mathbf{c}_l\|^2$ has a central chi-square distribution with $2N_r$ degrees of freedom. Therefore,

$$\mathbb{P}\left\{\|\mathbf{H}\mathbf{c}_l\|^2 < \frac{2^R - 1}{\gamma_0}\right\} \\ = 1 - \exp\left(-\frac{2^R - 1}{\gamma_0}\right) \underbrace{\sum_{m=0}^{N_r-1} \frac{1}{m!} \left(\frac{2^R - 1}{\gamma_0}\right)^m}_{:= f(\gamma_0)}, \quad l = 1, \dots, K. \quad (22)$$

Substituting (22) into (20) and (21), one obtains $t_{\min} K f(\gamma_0) \leq P_{\text{out}} \leq K f(\gamma_0)$. Then, using the squeezing theorem, we get the diversity gain

$$\lim_{\gamma_0 \rightarrow +\infty} -\frac{\log P_{\text{out}}}{\log \gamma_0} = \lim_{\gamma_0 \rightarrow +\infty} -\frac{\log f(\gamma_0)}{\log \gamma_0} = N_r. \quad (23)$$

The last equality in (23) can be derived by repeatedly applying L'Hospital's rule.

APPENDIX B PROOF OF A LEMMA

Lemma. If $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_{N_t}]$ is uniformly distributed on the group of $N_t \times N_t$ unitary matrices,

$$\mathbb{E}(\mathbf{u}_1 \mathbf{u}_1^H \mid \mathbf{u}_1 \in \mathcal{S}(\mathbf{c}_k)) = \frac{\alpha}{N_t} \mathbf{I}_{N_t} + (1 - \alpha) \mathbf{c}_k \mathbf{c}_k^H, \\ k = 1, \dots, K; \quad (24)$$

$$\mathbb{E}(\mathbf{u}_n \mathbf{u}_n^H \mid \mathbf{u}_1 \in \mathcal{S}(\mathbf{c}_k)) = \frac{N_t - \alpha}{N_t(N_t - 1)} \mathbf{I}_{N_t} - \frac{1 - \alpha}{N_t - 1} \mathbf{c}_k \mathbf{c}_k^H, \\ n = 2, \dots, N_t, \quad k = 1, \dots, K. \quad (25)$$

Proof: Since $\mathbf{U}$ is uniformly distributed, $\mathbf{u}_1$ is uniformly distributed on the unit hypersphere $\Omega^{N_t} = \{\mathbf{x} \in \mathbb{C}^{N_t} : \mathbf{x}^H \mathbf{x} = 1\}$. Conditioned on a particular realization of $\mathbf{u}_1$, $\mathbf{u}_n, n = 2, \dots, N_t$, is uniformly distributed on the set $\mathcal{O}(\mathbf{u}_1) = \{\mathbf{u}_n \in \Omega^{N_t} : \mathbf{u}_n^H \mathbf{u}_1 = 0\}$ [10]. Therefore

$$\mathbb{E}(\mathbf{u}_1 \mathbf{u}_1^H \mid \mathbf{u}_1 \in \mathcal{S}(\mathbf{c}_k)) = \int_{\mathcal{S}(\mathbf{c}_k)} C_0 \mathbf{u}_1 \mathbf{u}_1^H d\mathbf{u}_1, \quad (26)$$

$$\mathbb{E}(\mathbf{u}_n \mathbf{u}_n^H \mid \mathbf{u}_1 \in \mathcal{S}(\mathbf{c}_k)) = \int_{\mathcal{S}(\mathbf{c}_k)} C_0 \left(\int_{\mathcal{O}(\mathbf{u}_1)} C_1 \mathbf{u}_n \mathbf{u}_n^H d\mathbf{u}_n \right) d\mathbf{u}_1, \\ n = 2, \dots, N_t, \quad (27)$$

where $C_0 = \frac{(N_t - 1)!}{2\pi^{N_t} \alpha^{N_t - 1}}$, such that $\int_{\mathcal{S}(\mathbf{c}_k)} C_0 d\mathbf{u}_1 = 1$; and $C_1 = \frac{(N_t - 2)!}{2\pi^{N_t - 1}}$, such that $\int_{\mathcal{O}(\mathbf{u}_1)} C_1 d\mathbf{u}_n = 1$.

The calculation of (26). Let $\Theta = [\mathbf{c}_k; \Theta_0]$ be a unitary matrix, where Θ_0 is chosen arbitrarily with the constraint that Θ is unitary. Using the transformation $\mathbf{v} = \Theta^H \mathbf{u}_1$, $\mathcal{S}(\mathbf{c}_k)$ is rotated to $\mathcal{S}(\mathbf{e}_1) = \{\mathbf{x} \in \Omega^{N_t} : |\mathbf{x}^H \mathbf{e}_1|^2 > 1 - \alpha\}$ ($\mathbf{e}_1 = [1, 0, \dots, 0]^T$). The Jacobian for this transformation is 1, because Θ is unitary. Hence

$$\int_{\mathcal{S}(\mathbf{c}_k)} C_0 \mathbf{u}_1 \mathbf{u}_1^H d\mathbf{u}_1 = \int_{\mathcal{S}(\mathbf{e}_1)} C_0 \Theta \mathbf{v} \mathbf{v}^H \Theta^H d\mathbf{v} \\ = \Theta \left(C_0 \int_{\mathcal{S}(\mathbf{e}_1)} \mathbf{v} \mathbf{v}^H d\mathbf{v} \right) \Theta^H. \quad (28)$$

The surface integration $\int_{\mathcal{S}(\mathbf{e}_1)} \mathbf{v} \mathbf{v}^H d\mathbf{v}$ is converted to $2N_t - 1$ dimensional multiple integration [13]. Let

$$\mathcal{G} = \{\mathbf{y} \in \mathbb{R}^{2N_t - 1} : -\pi < y_1 < \pi, y_2^2 + \dots + y_{2N_t - 1}^2 < \alpha\}.$$

Construct the transformation $\mathbf{g} : \mathcal{G} \rightarrow \mathcal{S}(\mathbf{e}_1)$

$$\begin{cases} \Re(v_1) = g_1(\mathbf{y}) = \sqrt{1 - y_2^2 - \dots - y_{2N_t - 1}^2} \cos y_1 \\ \Im(v_1) = g_2(\mathbf{y}) = \sqrt{1 - y_2^2 - \dots - y_{2N_t - 1}^2} \sin y_1 \\ \Re(v_n) = g_{2n-1}(\mathbf{y}) = y_{2n-2}, & n = 2, \dots, N_t \\ \Im(v_n) = g_{2n}(\mathbf{y}) = y_{2n-1}, & n = 2, \dots, N_t \end{cases} \quad (29)$$

where $\Re(v_n)$ and $\Im(v_n)$ denote the real and imaginary parts of the n th entry of $\mathbf{v}$ respectively. It is easily verified that $\mathcal{S} \mathbf{g} = \sqrt{\det[(D\mathbf{g})^T (D\mathbf{g})]} = 1$, where $D\mathbf{g}$ is the differential matrix, with $[D\mathbf{g}]_{i,j} = \frac{\partial}{\partial y_j} g_i$, $i = 1, \dots, 2N_t$, $j = 1, \dots, 2N_t - 1$.

Then, under the transformation $\mathbf{v} = \mathbf{g}(\mathbf{y})$,

$$\begin{aligned}
& \left[\int_{\mathcal{S}(\mathbf{e}_1)} \mathbf{v} \mathbf{v}^H d\mathbf{v} \right]_{n,l} \\
&= \int_{\mathcal{S}(\mathbf{e}_1)} \left\{ [\Re(v_n)\Re(v_l) + \Im(v_n)\Im(v_l)] \right. \\
&\quad \left. + j[\Im(v_n)\Re(v_l) - \Re(v_n)\Im(v_l)] \right\} d\mathbf{v} \\
&= \int_{\mathcal{G}} \left\{ (g_{2n-1} g_{2l-1} + g_{2n} g_{2l}) \right. \\
&\quad \left. + j(g_{2n} g_{2l-1} - g_{2n-1} g_{2l}) \right\} \mathcal{J} \mathbf{g} d\mathbf{y} \\
&= \int_{\mathcal{G}} g_{2n-1} g_{2l-1} d\mathbf{y} + \int_{\mathcal{G}} g_{2n} g_{2l} d\mathbf{y} \\
&\quad + j \left(\int_{\mathcal{G}} g_{2n} g_{2l-1} d\mathbf{y} - \int_{\mathcal{G}} g_{2n-1} g_{2l} d\mathbf{y} \right) \quad (30)
\end{aligned}$$

The calculation of these integrations is straightforward, e.g. $\int_{\mathcal{G}} g_3 g_3 d\mathbf{y} = \int_{\mathcal{G}} y_2^2 d\mathbf{y} = \int_{-\pi}^{\pi} dy_1 \times \int_{y_2^2 + \dots + y_{2N_t-1}^2 < \alpha} y_2^2 dy_2 \dots dy_{2N_t-1} = \frac{\pi \alpha^{N_t}}{N_t!}$. Collecting the results, one obtains

$$C_0 \int_{\mathcal{S}(\mathbf{e}_1)} \mathbf{v} \mathbf{v}^H d\mathbf{v} = \frac{\alpha}{N_t} \mathbf{I}_{N_t} + (1-\alpha) \mathbf{e}_1 \mathbf{e}_1^H. \quad (31)$$

Substituting (31) into (28) and (26), we get (24).

The calculation of (27). The derivation of the inner integration in (27) is along the line of the calculation of (26) and omitted here. The result is

$$\int_{\mathcal{O}(\mathbf{u}_1)} C_1 \mathbf{u}_n \mathbf{u}_n^H d\mathbf{u}_n = \frac{1}{N_t-1} (\mathbf{I}_{N_t} - \mathbf{u}_1 \mathbf{u}_1^H). \quad (32)$$

Substituting (32) into (27), we get the result in (25), i.e.

$$\begin{aligned}
& \mathbb{E}(\mathbf{u}_n \mathbf{u}_n^H | \mathbf{u}_1 \in \mathcal{S}(\mathbf{c}_k)) \\
&= \frac{C_0}{N_t-1} \int_{\mathcal{S}(\mathbf{c}_k)} \mathbf{I}_{N_t} d\mathbf{u}_1 - \frac{C_0}{N_t-1} \int_{\mathcal{S}(\mathbf{c}_k)} \mathbf{u}_1 \mathbf{u}_1^H d\mathbf{u}_1 \\
&= \frac{N_t-\alpha}{N_t(N_t-1)} \mathbf{I}_{N_t} - \frac{1-\alpha}{N_t-1} \mathbf{c}_k \mathbf{c}_k^H. \quad (33)
\end{aligned}$$

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